

The Heyde Theorem on \mathbf{a} -adic Solenoids

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Abstract

We prove the following analogue of the Heyde theorem for \mathbf{a} -adic solenoids. Let ξ_1, ξ_2 be independent random variables taking values in an \mathbf{a} -adic solenoid $\Sigma_{\mathbf{a}}$ and with distributions μ_1, μ_2 . Let α_j, β_j be topological automorphisms of $\Sigma_{\mathbf{a}}$ such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1}$ are topological automorphisms of $\Sigma_{\mathbf{a}}$ too. Assuming that the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, we describe possible distributions μ_1, μ_2 .

KEY WORDS: Gaussian distribution, idempotent distribution, Heyde theorem, \mathbf{a} -adic solenoid
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1 Introduction

Many studies have been devoted to characterizing Gaussian distributions on the real line. Specifically, in 1970 Heyde proved the following theorem, which characterizes a Gaussian distribution by the symmetry of the conditional distribution of one linear form given another.

The Heyde theorem [Heyde ([12]; see also [13, Section 13.4.1])]. *Let ξ_1, \dots, ξ_n , $n \geq 2$, be independent random variables, α_j, β_j be nonzero constants such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ whenever $i \neq j$. If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ is symmetric then all random variables ξ_j are Gaussian.*

In recent years, a great deal of attention has been focused upon generalizing the classical characterization theorems to random variables with values in locally compact Abelian groups (see e.g. [1]–[4], [6]–[8], [14], [15]; see also [5], where one can find additional references). The articles [2]–[4], [14], [15] (see also [5, Chapter VI]) were devoted to finding group-theoretic analogues of the Heyde theorem. This article continues this research.

Let X be a second countable locally compact Abelian group, $\text{Aut}(X)$ be the group of topological automorphisms of X . Let ξ_j , $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables taking values in X and with distributions μ_j . Let $\alpha_j, \beta_j \in \text{Aut}(X)$ such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \text{Aut}(X)$ whenever $i \neq j$. Define the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$.

We formulate the following problem.

Problem 1. *Assume that conditional distribution of L_2 given L_1 is symmetric. Describe the possible distributions μ_j .*

Problem 1 was solved for the class of finite Abelian groups in [2], [14] and then for the class of countable discrete Abelian groups in [4], [15]. Problem 1 for \mathbf{a} -adic solenoids was formulated in the book [5]. In this article we solve this problem.

\mathbf{a} -adic solenoids are important examples of connected Abelian groups. We note that if X is a connected Abelian group of dimension one then X is topologically isomorphic either the real line \mathbb{R} , or the circle group \mathbb{T} , or an \mathbf{a} -adic solenoid $\Sigma_{\mathbf{a}}$. Problem 1 was solved for the case $X = \mathbb{R}$ by Heyde. Problem 1 cannot be formulated for the case $X = \mathbb{T}$ because there no exist

topological automorphisms α_j, β_j such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \text{Aut}(X)$ whenever $i \neq j$. In this article we solve Problem 1 (Theorem 1) for \mathbf{a} -adic solenoids $\Sigma_{\mathbf{a}}$. It turns out that the answer depends on topological automorphisms α_j, β_j . Note that it follows from [3] (see also [5, §16.2]) that under the condition that the characteristic functions of the distributions μ_j do not vanish the symmetry of the conditional distribution of L_2 given L_1 implies that μ_j are Gaussian.

2 Notation and definitions

Let X be a locally compact Abelian group, $Y = X^*$ be its character group, and (x, y) be the value of a character $y \in Y$ at an element $x \in X$. Let K be a subgroup of X . Denote by $A(Y, K) = \{y \in Y : (x, y) = 1 \text{ for all } x \in K\}$ the annihilator of K . If $\delta : X \rightarrow X$ is a continuous homomorphism, then the adjoint homomorphism $\tilde{\delta} : Y \rightarrow Y$ is defined by the formula $(x, \tilde{\delta}y) = (\delta x, y)$ for all $x \in X, y \in Y$. We note that $\delta \in \text{Aut}(X)$ if and only if $\tilde{\delta} \in \text{Aut}(Y)$. For each integer $n, n \neq 0$, let $f_n : X \rightarrow X$ be the homomorphism $f_n x = nx$. Set $X^{(n)} = f_n(X), X_{(n)} = \text{Ker } f_n$. It is clear that the adjoint homomorphism $\tilde{f}_n : Y \rightarrow Y$ is the mapping $\tilde{f}_n y = ny$. Denote by \mathbb{R} the additive group of real numbers, by \mathbb{Z} the group of integers, by \mathbb{Q} the group of rational numbers considering in the discrete topology, by $\mathbb{Z}(n)$ the finite cyclic group of order n . For a fixed prime p denote by $\mathbb{Z}(p^\infty)$ the set of rational numbers of the form $\{k/p^n : k = 0, 1, \dots, p^n - 1, n = 0, 1, \dots\}$ and define the operation in $\mathbb{Z}(p^\infty)$ as addition modulo 1. Then $\mathbb{Z}(p^\infty)$ is transformed into an Abelian group, which we consider in the discrete topology. Denote by $\text{Aut}(X)$ the group of topological automorphisms of the group X .

Put $\mathbf{a} = (a_0, a_1, \dots)$, where all $a_j \in \mathbb{Z}, a_j > 1$. First we recall the definition of the group of \mathbf{a} -adic integers $\Delta_{\mathbf{a}}$ [10, (10.2)]. As a set $\Delta_{\mathbf{a}}$ coincides with the Cartesian product $\prod_{n=0}^{\infty} \{0, 1, \dots, a_n - 1\}$. For $\mathbf{x} = (x_0, x_1, x_2, \dots), \mathbf{y} = (y_0, y_1, y_2, \dots) \in \Delta_{\mathbf{a}}$ let $\mathbf{z} = \mathbf{x} + \mathbf{y}$ be defined as follows. Let $x_0 + y_0 = t_0 a_0 + z_0$, where $z_0 \in \{0, 1, \dots, a_0 - 1\}, t_0 \in \{0, 1\}$. Assume that the numbers $z_0, z_1, \dots, z_k; t_0, t_1, \dots, t_k$ have been already determined. Let us put then $x_{k+1} + y_{k+1} + t_k = t_{k+1} a_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, \dots, a_{k+1} - 1\}, t_{k+1} \in \{0, 1\}$. This defines by induction a sequence $\mathbf{z} = (z_0, z_1, z_2, \dots)$. The set $\Delta_{\mathbf{a}}$ with the addition defined above is an Abelian group, whose neutral element is the sequence in $\Delta_{\mathbf{a}}$ that is identically zero. Consider $\Delta_{\mathbf{a}}$ in the product topology. The obtained group is called the \mathbf{a} -adic integers. If all of the integers a_j are equal to some fixed prime integer p , we write Δ_p instead of $\Delta_{\mathbf{a}}$, and call this object the group of p -adic integers. Note that $\Delta_p^* \approx \mathbb{Z}(p^\infty)$ (see [10, (25.2)]).

Consider the group $\mathbb{R} \times \Delta_{\mathbf{a}}$. Let B be the subgroup of the group $\mathbb{R} \times \Delta_{\mathbf{a}}$ of the form $B = \{(n, n\mathbf{u})\}_{n=-\infty}^{\infty}$, where $\mathbf{u} = (1, 0, \dots, 0, \dots)$. The factor-group $\Sigma_{\mathbf{a}} = (\mathbb{R} \times \Delta_{\mathbf{a}})/B$ is called the \mathbf{a} -adic solenoid. The group $\Sigma_{\mathbf{a}}$ is compact and connected, and has dimension one ([10, (10.12), (10.13), (24.28)]). The character group of the group $\Sigma_{\mathbf{a}}$ is topologically isomorphic to the subgroup $H_{\mathbf{a}} \subset \mathbb{Q}$ of the form

$$H_{\mathbf{a}} = \left\{ \frac{m}{a_0 a_1 \dots a_n} : n = 0, 1, \dots; m \in \mathbb{Z} \right\}.$$

We will assume without loss of generality that if $X = \Sigma_{\mathbf{a}}$ then $Y = X^* = H_{\mathbf{a}}$.

Let Y be an Abelian group, $f(y)$ be a function on Y , and $h \in Y$. Denote by Δ_h the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y).$$

A function $f(y)$ on Y is called a polynomial if

$$\Delta_h^{n+1} f(y) = 0$$

for some n and for all $y, h \in Y$. If Y is a subgroup of \mathbb{Q} then this definition of a polynomial coincides with the classical one.

Let $M^1(X)$ be the convolution semigroup of probability distributions on X , $\hat{\mu}(y) = \int_X(x, y)d\mu(x)$ be the characteristic function of a distribution $\mu \in M^1(X)$, and $\sigma(\mu)$ be the support of μ . If K is a closed subgroup of X and $\sigma(\mu) \subset K$, then $\hat{\mu}(y + h) = \hat{\mu}(y)$ for all $y \in Y$, $h \in A(Y, K)$. If E is a closed subgroup of Y and $\hat{\mu}(y) = 1$ for $y \in E$, then $\hat{\mu}(y + h) = \hat{\mu}(y)$ for all $y \in Y$, $h \in E$ and $\sigma(\mu) \subset A(X, E)$. For $\mu \in M^1(X)$ we define the distribution $\bar{\mu} \in M^1(X)$ by the rule $\bar{\mu}(B) = \mu(-B)$ for all Borel sets $B \subset X$. Observe that $\hat{\bar{\mu}}(y) = \overline{\hat{\mu}(y)}$.

A distribution $\gamma \in M^1(X)$ is called Gaussian ([16, §4.6]) if its characteristic function can be represented in the form

$$\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\},$$

where $x \in X$ and $\varphi(y)$ is a continuous nonnegative function satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y. \quad (1)$$

Denote by $\Gamma(X)$ the set of Gaussian distributions on X . It is easy to see that any nonnegative function $\varphi(y)$ on the group $H_{\mathbf{a}}$ satisfying equation (1) is of the form $\varphi(y) = \lambda y^2$, where $\lambda \geq 0$, $y \in H_{\mathbf{a}}$. It is well-known that a support of a Gaussian distribution on a locally compact Abelian group X is a coset of a connected subgroup of X . Thus if γ is a non degenerate Gaussian distribution on $X = \Sigma_{\mathbf{a}}$ then $\sigma(\gamma) = X$.

Denote by $I(X)$ the set of idempotent distributions on X , i.e. the set of shifts of Haar distributions m_K of compact subgroups K of the group X . Observe that the characteristic function of the Haar distribution m_K is of the form

$$\hat{m}_K(y) = \begin{cases} 1, & y \in A(Y, K); \\ 0, & y \notin A(Y, K). \end{cases} \quad (2)$$

We note that if a distribution $\mu \in \Gamma(X) * I(X)$, i.e. $\mu = \gamma * m_K$, where $\gamma \in \Gamma(X)$, then μ is invariant with respect to a compact subgroup $K \subset X$ and under the natural homomorphism $X \mapsto X/K$ μ induces a Gaussian distribution on the factor group X/K .

3 The Heyde theorem (the general case)

Let ξ_1, ξ_2 be independent random variables with values in the group $X = \Sigma_{\mathbf{a}}$ and distributions μ_1, μ_2 . Consider the linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$, where $\alpha_j, \beta_j \in \text{Aut}(X)$ and $\beta_1 \alpha_1^{-1} \pm \beta_2 \alpha_2^{-1} \in \text{Aut}(X)$. Assume that the conditional distribution of linear form L_2 given L_1 is symmetric. Taking into consideration new independent random variables $\xi'_j = \alpha_j \xi_j$ we reduce the study of the distributions μ_j on X to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1 \xi_1 + \delta_2 \xi_2$, where $\delta_j \in \text{Aut}(X)$ and $\delta_1 \pm \delta_2 \in \text{Aut}(X)$. Note that any topological automorphism δ of the group X is of the form

$$\delta = f_p f_q^{-1}$$

for some relatively prime p and q , where $f_p, f_q \in \text{Aut}(X)$. Note that for any $\delta \in \text{Aut}(X)$ the conditional distribution of the linear form L_2 given L_1 is symmetric if and only if the conditional distribution of the linear form δL_2 given L_1 is symmetric. Hence without loss of generality, we can assume from the beginning that $L_1 = \xi_1 + \xi_2$, $L_2 = p\xi_1 + q\xi_2$, where $p, q \in \mathbb{Z}$, $pq \neq 0$, p and q are relatively prime, $f_p, f_q, f_{p \pm q} \in \text{Aut}(X)$. Now we formulate the main result of this article.

Theorem 1. *Let $X = \Sigma_{\mathbf{a}}$. Assume that $f_p, f_q, f_{p \pm q} \in \text{Aut}(X)$, p and q are relatively prime. The following statements hold:*

1. Assume that $pq = -3$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 . If the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric then at least one distribution $\mu_j \in \Gamma(X) * I(X)$.
2. Assume that $pq \neq -3$. Then there exist independent random variables ξ_1, ξ_2 with values in X and distributions μ_1, μ_2 such that the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric and the distributions $\mu_j \notin \Gamma(X) * I(X)$, $j = 1, 2$.

Theorem 1 can be regarded as a group analogue of the Heyde theorem for \mathbf{a} -adic solenoids. To prove Theorem 1 we need some lemmas.

Lemma 1. *Let X be a locally compact second countable Abelian group. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 . Consider the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$, where α_j, β_j are continuous homomorphisms of the group X . The conditional distribution of the linear form L_2 given L_1 is symmetric if and only if the characteristic functions of the distributions μ_j satisfy the equation*

$$\hat{\mu}_1(\tilde{\alpha}_1 u + \tilde{\beta}_1 v) \hat{\mu}_2(\tilde{\alpha}_2 u + \tilde{\beta}_2 v) = \hat{\mu}_1(\tilde{\alpha}_1 u - \tilde{\beta}_1 v) \hat{\mu}_2(\tilde{\alpha}_2 u - \tilde{\beta}_2 v), \quad u, v \in Y. \quad (3)$$

Lemma 1 was proved in [5, §16.1] in the case where $\alpha_j, \beta_j \in \text{Aut}(X)$. This proof is valid for arbitrary continuous homomorphisms α_j, β_j of the group X .

Lemma 2. *Let either $|q| = 2$ or $q = 4m + 3$, where m is some integer. Let $X = \Delta_2$. Then there exist independent identically distributed random variables ξ_1, ξ_2 with values in X and distribution $\mu \notin I(X)$ such that the conditional distribution of the linear form $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.*

Proof. Since $X = \Delta_2$, we have $Y \approx \mathbb{Z}(2^\infty)$. To avoid introducing new notation we will assume that $Y = \mathbb{Z}(2^\infty)$.

Let $g_0(y)$ be an arbitrary characteristic functions on $Y_{(2)}$. Set

$$g(y) = \begin{cases} g_0(y), & y \in Y_{(2)}; \\ 0, & y \notin Y_{(2)}. \end{cases}$$

The function $g(y)$ is a positive definite function on Y ([11, §32]). By the Bochner theorem there exists a distribution $\mu \in M^1(X)$ such that $\hat{\mu}(y) = g(y)$. It is clear that $g_0(y)$ can be chosen in such a way that $\mu \notin I(X)$.

Let ξ_1, ξ_2 be independent identically distributed random variables with values in X and distribution μ . We check that the conditional distribution of the linear form $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. By Lemma ?? it suffices to show that the characteristic function $\hat{\mu}(y)$ satisfies equation (3) which takes the form

$$\hat{\mu}(u + v) \hat{\mu}(u + qv) = \hat{\mu}(u - v) \hat{\mu}(u - qv), \quad u, v \in Y. \quad (4)$$

Since $Y_{(2)} \approx \mathbb{Z}(2)$, it is clear that if $u, v \in Y_{(2)}$ then equation (4) is an equality.

If either $u \in Y_{(2)}, v \notin Y_{(2)}$ or $u \notin Y_{(2)}, v \in Y_{(2)}$ then $u \pm v \notin Y_{(2)}$. Hence $\hat{\mu}(u + v) = \hat{\mu}(u - v) = 0$ and equation (4) is an equality.

Let $u, v \notin Y_{(2)}$. Suppose that the left-hand side of equation (4) does not vanish. Then

$$u + v \in Y_{(2)}, \quad u + qv \in Y_{(2)}. \quad (5)$$

Let $q = 2$. It follows from (5) that $v \in Y_{(2)}$, contrary to the choice of v . Hence the left-hand side of equation (4) is equal to zero. Similarly, we prove that the right-hand side of equation (4) is equal to zero.

Let $q = -2$. It follows from (5) that $3v \in Y_{(2)}$. Since $f_3 \in \text{Aut}(Y)$ and $Y_{(2)}$ is a characteristic subgroup, we have $v \in Y_{(2)}$, contrary to the choice of v . Hence the left-hand side of equation (4) is equal to zero. Similarly, we prove that the right-hand side of equation (4) is equal to zero.

Let $q = 4m + 3$. It follows from (5) that $(q - 1)v \in Y_{(2)}$. Since $(q - 1) = 2(2m + 1)$ and $f_{2m+1} \in \text{Aut}(Y)$, we have $2v \in Y_{(2)}$. Hence v is an element of order 4. So, $qv = -v$. It follows from this that equation (4) is an equality. Assume now that the right-hand side of equation (4) does not vanish. Similarly, we prove that in this case equation (4) is an equality.

Lemma 3. *Let $q = 4m + 1$ where $m \notin \{0, -1\}$. Let $|2m + 1| = p_1^{l_1} \times \cdots \times p_k^{l_k}$ — be a decomposition of $|2m + 1|$ into prime factors. Let $X = \Delta_{p_1} \times \cdots \times \Delta_{p_k}$. Then there exist independent identically distributed random variables ξ_1, ξ_2 with values in X and distribution $\mu \notin I(X)$ such that the conditional distribution of the linear form $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.*

Proof. Since $X = \Delta_{p_1} \times \cdots \times \Delta_{p_k}$, we have $Y \approx \mathbb{Z}(p_1^\infty) \times \cdots \times \mathbb{Z}(p_k^\infty)$. To avoid introducing new notation we will assume that $Y = \mathbb{Z}(p_1^\infty) \times \cdots \times \mathbb{Z}(p_k^\infty)$.

Let $g_0(y)$ be an arbitrary characteristic functions on $Y_{(2m+1)}$. Set

$$g(y) = \begin{cases} g_0(y), & y \in Y_{(2m+1)}; \\ 0, & y \notin Y_{(2m+1)}. \end{cases}$$

The function $g(y)$ is a positive definite function on Y ([11, §32]). By the Bochner theorem there exists a distribution $\mu \in M^1(X)$ such that $\hat{\mu}(y) = g(y)$, $j = 1, 2$. It is clear that $g_0(y)$ can be chosen in such a way that $\mu \notin I(X)$.

Let ξ_1, ξ_2 be independent identically distributed random variables with values in X and distribution μ . We check that the conditional distribution of the linear form $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. By Lemma 1 it suffices to show that the characteristic function $\hat{\mu}(y)$ satisfies equation (3) which takes the form (4).

Let $u, v \in Y_{(2m+1)}$. Then $qv = (q + 1)v - v = -v$ and equation (4) is an equality.

If either $u \in Y_{(2m+1)}, v \notin Y_{(2m+1)}$ or $u \notin Y_{(2m+1)}, v \in Y_{(2m+1)}$, then $u \pm v \notin Y_{(2m+1)}$. Hence $\hat{\mu}(u + v) = \hat{\mu}(u - v) = 0$ and equation (4) is an equality.

Let $u, v \notin Y_{(2m+1)}$. Suppose that the left-hand side of equation (4) does not vanish. Then $u + v \in Y_{(2m+1)}, u + qv \in Y_{(2m+1)}$. Hence $(q - 1)v \in Y_{(2m+1)}$. Since $q - 1 = 4m$ and $f_{4m} \in \text{Aut}(Y)$, we have $v \in Y_{(2m+1)}$, contrary to the choice of v . Hence the left-hand side of equation (4) is equal to zero. Similarly, we prove that the right-hand side of equation (4) is equal to zero.

Lemma 4. *Let $X = \Sigma_{\mathbf{a}}$. If $f_n \in \text{Aut}(X)$, where $n = p_1^{l_1} \times \cdots \times p_k^{l_k}$ is a decomposition of n into prime factors, then the group X contains a subgroup topologically isomorphic to $\Delta_{p_1} \times \cdots \times \Delta_{p_k}$.*

Proof. Since $X = \Sigma_{\mathbf{a}}$, the character group $Y = H_{\mathbf{a}}$ is a subgroup of \mathbb{Q} . As is well known that

$$\mathbb{Q}/\mathbb{Z} \approx \prod_{p \in \mathcal{P}} \mathbb{Z}(p^\infty),$$

where \mathcal{P} is the set of prime numbers ([9, §8]). Since $Y \subset \mathbb{Q}$, we have $Y/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. The condition $f_n \in \text{Aut}(X)$ implies that all $f_{p_j} \in \text{Aut}(X)$. Hence $f_{p_j} \in \text{Aut}(Y)$. It is obvious that if p is a prime number and $f_p \in \text{Aut}(Y)$ then $F_p \subset Y/\mathbb{Z}$, where $F_p \approx \mathbb{Z}(p^\infty)$. Hence $L \subset Y/\mathbb{Z}$, where $L \approx \mathbb{Z}(p_1^\infty) \times \cdots \times \mathbb{Z}(p_k^\infty)$. It is clear that $Y/\mathbb{Z} = L \times M$, where M is a group. Since $(Y/\mathbb{Z})^* \approx A(X, \mathbb{Z}) \subset X$ and $(Y/\mathbb{Z})^* \approx L^* \times M^*$, the group X contains a subgroup topologically isomorphic to $L^* \times M^*$. The statement of Lemma 4 follows from the form of L .

Lemma 5. *Let X be a locally compact second countable Abelian group. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 . Consider the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$, where α_j, β_j are continuous homomorphisms of the group X . The linear forms L_1 and L_2 are independent if and only if the characteristic functions of the distributions μ_j satisfy the equation*

$$\hat{\mu}_1(\tilde{\alpha}_1 u + \tilde{\beta}_1 v) \hat{\mu}_2(\tilde{\alpha}_2 u + \tilde{\beta}_2 v) = \hat{\mu}_1(\tilde{\alpha}_1 u) \hat{\mu}_1(\tilde{\beta}_1 v) \hat{\mu}_2(\tilde{\alpha}_2 u) \hat{\mu}_2(\tilde{\beta}_2 v), \quad u, v \in Y. \quad (6)$$

Lemma 5 was proved in [5, §10.1] in the case where $\alpha_j, \beta_j \in \text{Aut}(X)$. This proof is valid for arbitrary continuous homomorphisms α_j, β_j of the group X .

Lemma 6. *Let X be a locally compact second countable Abelian group, δ_1, δ_2 be continuous homomorphisms of the group X . Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 . If the conditional distribution of the linear form $L_2 = \delta_1\xi_1 + \delta_2\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric then the linear forms $L'_1 = (\delta_1 + \delta_2)\xi_1 + 2\delta_2\xi_2$ and $L'_2 = 2\delta_1\xi_1 + (\delta_1 + \delta_2)\xi_2$ are independent.*

Proof. By Lemma 1 the symmetry of the conditional distribution of the linear form L_2 given L_1 implies that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation

$$\hat{\mu}_1(u + \varepsilon_1 v) \hat{\mu}_2(u + \varepsilon_2 v) = \hat{\mu}_1(u - \varepsilon_1 v) \hat{\mu}_2(u - \varepsilon_2 v), \quad u, v \in Y, \quad (7)$$

where $\varepsilon_j = \tilde{\delta}_j$.

Putting $u = \varepsilon_2 y, v = -y$ and then $u = -\varepsilon_1 y, v = y$ into equation (7) we obtain

$$\hat{\mu}_1((\varepsilon_2 - \varepsilon_1)y) = \hat{\mu}_1((\varepsilon_1 + \varepsilon_2)y) \hat{\mu}_2(2\varepsilon_2 y), \quad y \in Y, \quad (8)$$

$$\hat{\mu}_2((\varepsilon_2 - \varepsilon_1)y) = \hat{\mu}_1(-2\varepsilon_1 y) \hat{\mu}_2(-(\varepsilon_1 + \varepsilon_2)y), \quad y \in Y. \quad (9)$$

Let $t, s \in Y$. Putting $u = \varepsilon_1 s + \varepsilon_2 t, v = s + t$ into (7) we obtain

$$\hat{\mu}_1((\varepsilon_1 + \varepsilon_2)t + 2\varepsilon_1 s) \hat{\mu}_2(2\varepsilon_2 t + (\varepsilon_1 + \varepsilon_2)s) = \hat{\mu}_1((\varepsilon_2 - \varepsilon_1)t) \hat{\mu}_2(-(\varepsilon_2 - \varepsilon_1)s), \quad s, t \in Y. \quad (10)$$

Taking into account (8) and (9) equation (10) can be written in the form

$$\begin{aligned} & \hat{\mu}_1((\varepsilon_1 + \varepsilon_2)t + 2\varepsilon_1 s) \hat{\mu}_2(2\varepsilon_2 t + (\varepsilon_1 + \varepsilon_2)s) = \\ & \hat{\mu}_1((\varepsilon_1 + \varepsilon_2)t) \hat{\mu}_2(2\varepsilon_2 t) \hat{\mu}_1(2\varepsilon_1 s) \hat{\mu}_2((\varepsilon_1 + \varepsilon_2)s), \quad s, t \in Y. \end{aligned} \quad (11)$$

Lemma 5 and equation (3) imply that the linear forms $L'_1 = (\delta_1 + \delta_2)\xi_1 + 2\delta_2\xi_2$ and $L'_2 = 2\delta_1\xi_1 + (\delta_1 + \delta_2)\xi_2$ are independent.

Remark 1. Lemma 6 implies that the Heyde theorem on the group \mathbb{R} for $n = 2$ can be obtained from the Skitovich-Darmois theorem.

Proof of Theorem 1. By Lemma 1 the symmetry of the conditional distribution of the linear form L_2 given L_1 implies that the characteristic functions of distributions μ_j satisfy equation (3) which takes the form

$$\hat{\mu}_1(u + pv) \hat{\mu}_2(u + qv) = \hat{\mu}_1(u - pv) \hat{\mu}_2(u - qv), \quad u, v \in Y. \quad (12)$$

We will study the solutions of this equation.

Consider first the case where $pq = -3$. Obviously, without loss of generality we can assume that $p = 1$ and $q = -3$ that is $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - 3\xi_2$. Lemma 6 implies that the linear forms $L'_1 = -2\xi_1 - 6\xi_2$ and $L'_2 = 2\xi_1 - 2\xi_2$ are independent. Making the substitution $\zeta_1 = 2\xi_1$ and $\zeta_2 = -2\xi_2$, we obtain that the linear forms $L''_1 = -\zeta_1 + 3\zeta_2$ and $L''_2 = \zeta_1 + \zeta_2$ are also independent. As has been proved in [6] the independence of the linear forms L''_1 and L''_2 implies that at least the distribution of one random variable ζ_j belongs to $\Gamma(X) * I(X)$. Returning to the random variables ξ_j , we obtain the statement 1 of Theorem 1.

Consider now the case where $pq \neq -3$. Two cases are possible: pq is a composite number and pq is a prime number.

We prove that in these cases there exist independent random variables ξ_1 and ξ_2 with values in X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ such that the conditional distribution of the linear form L_2 given L_1 is symmetric.

Case 1. pq is a composite number. In this case we follow the scheme of the proof of the analogous case in Theorem 1 of the article [6].

Put $s = p - q$, and decompose $|s|$ into prime factors $|s| = s_1^{k_1} \dots s_l^{k_l}$. Denote by H the subgroup of Y of the form

$$H = \left\{ \frac{m}{s_{j_1}^{n_1} \dots s_{j_r}^{n_r}} : m, n_j \in \mathbb{Z} \right\}.$$

If $|s| = 1$ we suppose that $H = \mathbb{Z}$. Set $G = H^*$.

1a. $|p| > 1, |q| > 1$.

Since p and s are relatively prime, and so q and s , we have $H^{(p)} \neq H$ and $H^{(q)} \neq H$. Assume that $\lambda_j \in M^1(G)$ and $\sigma(\lambda_1) \subset A(G, H^{(p)})$, $\sigma(\lambda_2) \subset A(G, H^{(q)})$. It follows from this that $\hat{\lambda}_1(y) = 1$, $y \in H^{(p)}$, and $\hat{\lambda}_2(y) = 1$, $y \in H^{(q)}$. Therefore

$$\hat{\lambda}_1(u + pv) = \hat{\lambda}_1(u), \quad \hat{\lambda}_2(u + qv) = \hat{\lambda}_2(u), \quad u, v \in H. \quad (13)$$

Consider the functions $g_j(y)$ on the group Y of the form

$$g_j(y) = \begin{cases} \hat{\lambda}_j(y), & y \in H; \\ 0, & y \notin H. \end{cases} \quad (14)$$

The functions $g_j(y)$ are positive definite functions on Y ([11, §32]). By the Bochner theorem there exist distributions $\mu_j \in M^1(X)$ such that $\hat{\mu}_j(y) = g_j(y)$, $j = 1, 2$. We will show that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (12).

We conclude from (13) and (14) that if $u, v \in H$, then equation (12) is an equality.

Let either $u \in H, v \notin H$ or $u \notin H, v \in H$. Since the numbers p and s are relatively prime, we have either $pv \notin H$ or $pv \in H$ respectively. So, $u \pm pv \notin H$ and hence $\hat{\mu}_1(u + pv) = \hat{\mu}_1(u - pv) = 0$, and equation (12) is an equality.

Let $u, v \notin H$. Suppose that the left-hand side of equation (12) does not vanish. Then $u + pv \in H$ and $u + qv \in H$. Hence $sv \in H$. Therefore $v \in H$, contrary to the choice of v . Hence the left-hand side of (12) is equal to zero. Similarly, we prove that the right-hand side of (12) is equal to zero.

So, the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (12). If ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j , then by Lemma 1 the conditional

distribution of the linear form L_2 given L_1 is symmetric. It is clear that λ_j can be chosen in such a way that $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$. The desired statement in case **1a** is proven.

1b. Either $|p| = 1, |q| > 1$ or $|p| > 1, |q| = 1$.

Assume for definiteness that $|p| = 1$. Without loss of generality, we suppose $p = 1$. Let $q = q_1 q_2$ be a decomposition of q , where $|q_j| > 1, j = 1, 2$. It is obvious that if $f_q \in \text{Aut}(X)$, then $f_{q_1}, f_{q_2} \in \text{Aut}(X)$. Note that the conditional distribution of $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric if and only if the conditional distribution of $L_2 = \frac{1}{q_1}\xi_1 + q_2\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. Making the substitution $\zeta_1 = \frac{1}{q_1}\xi_1$, we reduce the problem to the case when $L_1 = q_1\xi_1 + \xi_2, L_2 = \xi_1 + q_2\xi_2$. Equation (12) in this case takes the form

$$\hat{\mu}_1(q_1u + v)\hat{\mu}_2(u + q_2v) = \hat{\mu}_1(q_1u - v)\hat{\mu}_2(u - q_2v), \quad u, v \in Y. \quad (15)$$

Assume that $\lambda_j \in M^1(G)$ and $\sigma(\lambda_j) \subset A(G, H^{(q_j)}), j = 1, 2$. It is obvious that $\hat{\lambda}_j(y) = 1, y \in H^{(q_j)}$. Hence

$$\hat{\lambda}_1(q_1u + v) = \hat{\lambda}_1(v), \quad \hat{\lambda}_2(u + q_2v) = \hat{\lambda}_2(u) \quad u, v \in H. \quad (16)$$

In the same manner as in case **1a** we define the functions $g_j(y)$ by formulas (14) and distributions $\mu_j \in M^1(X)$. We will show that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (15).

We conclude from (16) and (14) that if $u, v \in H$, then equation (15) is an equality.

Let either $u \in H, v \notin H$ or $u \notin H, v \in H$. Since the numbers q_1 and s are relatively prime, we have either $pu \in H$ or $pu \notin H$ respectively. So, $q_1u \pm v \notin H$ and hence $\hat{\mu}_1(q_1u + v) = \hat{\mu}_1(q_1u - v) = 0$, and equation (15) is an equality.

Let $u, v \notin H$. Suppose that the left-hand side of equation (15) does not vanish. Then $q_1u + v \in H$ and $u + q_2v \in H$. Hence $su \in H$. Therefore $u \in H$, contrary to the choice of u . Hence the left-hand side of equation (15) is equal to zero. Reasoning similarly we show that the right-hand side of (15) is equal to zero.

So, the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (15). If ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j , then by Lemma 1 the conditional distribution of the linear form L_2 given L_1 is symmetric. It is clear that λ_j can be chosen in such a way that $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$. The desired statement in case **1b** is proven.

Case 2. pq is a prime number, i.e. either $|p| = 1, |q| > 1$ or $|p| > 1, |q| = 1$.

Assume for definiteness that $p = 1$ and q is a prime number, i.e. $L_1 = \xi_1 + \xi_2, L_2 = \xi_1 + q\xi_2$. Equation (12) takes the form

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u + qv) = \hat{\mu}_1(u - v)\hat{\mu}_2(u - qv), \quad u, v \in Y. \quad (17)$$

2a. $|q| = 2$.

Since $f_2 \in \text{Aut}(X)$, Lemma 4 implies that the group X contains a subgroup topologically isomorphic to Δ_2 . Then the statement 2 of Theorem 1 follows from Lemma 2.

Let q be an odd number. There exist two possibilities: 1) $q = 4m + 3$; 2) $q = 4m + 1$.

2b. $q = 4m + 3$.

Note that since q is an odd number and $f_{q+1} \in \text{Aut}(X)$, the homomorphism $f_2 \in \text{Aut}(X)$. Hence Lemma 4 implies that the group X contains a subgroup topologically isomorphic to Δ_2 . Then the statement 2 of Theorem 1 follows from Lemma 2.

2c. $q = 4m + 1$ ($m \neq -1$).

Since $f_{q+1} \in \text{Aut}(X)$ and $q + 1 = 2(2m + 1)$, the homomorphism $f_{2m+1} \in \text{Aut}(X)$. Let $|2m + 1| = p_1^{l_1} \times \dots \times p_k^{l_k}$ be a decomposition of $|2m + 1|$ into prime factors. Lemma 4 implies that the group X contains a subgroup topologically isomorphic to $\Delta_{p_1} \times \dots \times \Delta_{p_k}$. Then the statement 2 of Theorem 1 follows from Lemma 3.

Remark 2. Statement 1 of Theorem 1 may not be strengthened. Namely, the following statement is valid. If $pq = -3$ then there exist independent random variables ξ_1, ξ_2 with values in X and distributions μ_1, μ_2 such that the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric and one of the distributions $\mu_j \notin \Gamma(X) * I(X)$.

It suffices to consider the case when $p = 1, q = -3$. We shall construct solutions of equation (3) which takes the form

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u - 3v) = \hat{\mu}_1(u - v)\hat{\mu}_2(u + 3v), \quad u, v \in Y. \quad (18)$$

Let γ_1 and γ_2 be Gaussian distributions on X with characteristic functions $\hat{\gamma}_1(y) = e^{-3y^2}$ and $\hat{\gamma}_2(y) = e^{-y^2}$. It is easy to verify that these functions satisfy equation (18).

Since $f_{p+q} \in \text{Aut}(X)$, we have $f_2 \in \text{Aut}(X)$. Hence $f_2 \in \text{Aut}(Y)$. Therefore the group Y contains a subgroup of dyadic rational numbers. Denote by H this subgroup. Since $f_q \in \text{Aut}(X)$, we have $f_3 \in \text{Aut}(X)$. Hence $f_3 \in \text{Aut}(Y)$. Therefore $1/3 \in Y$. Denote by L a subgroup in Y which is generated by the subgroup H and the element $1/3$. Observe that $L = \{H, 1/3 + H, 2/3 + H\}$. Let $G = A(X, H)$, $K = A(X, L)$. Let $\omega_1 = (1/2)[m_G + m_K]$ and $\omega_2 = m_G$. It follows from (2) that

$$\hat{\omega}_1(y) = \begin{cases} 1, & y \in H; \\ 1/2, & y \in L \setminus H; \\ 0, & y \notin L. \end{cases} \quad \hat{\omega}_2(y) = \begin{cases} 1, & y \in H; \\ 0, & y \notin H. \end{cases} \quad (19)$$

We verify that these functions satisfy equation (19).

It is clear that if $u \in H, v \in L$ then equation (19) is an equality.

If $u \in L \setminus H, v \in L$ then $u \pm 3v \notin H$. Hence $\hat{\omega}_2(u - 3v) = \hat{\omega}_2(u + 3v) = 0$ and equation (19) is an equality.

If either $u \in L, v \notin L$ or $u \notin L, v \in L$ then $u \pm v \notin L$. Hence $\hat{\omega}_1(u + v) = \hat{\omega}_2(u - v) = 0$ and equation (19) is an equality.

Let $u, v \notin L$. Suppose that the left-hand side of equation (19) does not vanish. Then $u + v \in L$ and $u - 3v \in H$. Hence $4v \in L$. We obtain that $v \in L$, contrary to the choice of v . Hence the left-hand side of equation (19) is equal to zero. Reasoning similarly we show that the right-hand side of equation (19) is equal to zero.

Put $\mu_j = \gamma_j * \omega_j, j = 1, 2$. It is obvious that the functions $\hat{\mu}_j(y)$ satisfy equation (19). If ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j then Lemma 1 implies that the conditional distribution of the linear form $L_2 = \xi_1 - 3\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. By the construction $\mu_1 \notin \Gamma(X) * I(X)$ and $\mu_2 \in \Gamma(X) * I(X)$.

Remark 3. Note that in Theorem 1 we suppose that there exist for some relatively prime p and q automorphisms f_p and f_q such that $f_{p \pm q} \in \text{Aut}(X)$ on the group $X = \Sigma_{\mathbf{a}}$. It is easy to prove that groups $X = \Sigma_{\mathbf{a}}$ have this property if and only if $f_2, f_3 \in \text{Aut}(X)$.

Remark 4. We note that if in Theorem 1 distributions μ_1, μ_2 have non-vanishing characteristic functions, then $\mu_1, \mu_2 \in \Gamma(X)$. Indeed, it follows from conditions on coefficients of the linear forms that one of numbers $p, q, p \pm q$ is even. So, $f_2 \in \text{Aut}(X)$. Hence the group $X = \Sigma_{\mathbf{a}}$

does not contain elements of order two. The following theorem (see [3]) implies the desired statement: Let X be a locally compact second countable Abelian group containing no elements of order two. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 with nonvanishing characteristic functions. Consider the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1 \xi_1 + \delta_2 \xi_2$, where $\delta_j, \delta_1 \pm \delta_2 \in \text{Aut}(X)$. If the conditional distribution of the linear form L_2 given L_1 is symmetric then $\mu_1, \mu_2 \in \Gamma(X)$.

4 The Heyde theorem (the special case)

We prove in this section that Theorem 1 can be essentially strengthened if we assume in addition that the support at least one of distributions μ_j is not contained in a coset of a proper closed subgroup of the group X .

Let $\mu \in M^1(X)$. It is easy to see that μ has the property: $\sigma(\mu)$ is not contained in a coset of a proper closed subgroup of the group X if and only if

$$\{y \in Y : |\hat{\mu}(y)| = 1\} = \{0\}. \quad (20)$$

Theorem 2. *Let $X = \Sigma_{\mathbf{a}}$. Assume that $f_p, f_q, f_{p \pm q} \in \text{Aut}(X)$, p and q are relatively prime. The following statements hold:*

1. *Let $pq > 0$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 such that at least one support $\sigma(\mu_j)$ is not contained in a coset of a proper closed subgroup of the group X . If the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric then $\mu_1 = \mu_2 = m_X$.*
2. *Let $pq = -3$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 such that at least one support $\sigma(\mu_j)$ is not contained in a coset of a proper closed subgroup of the group X . If the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric then at least one distribution $\mu_j \in \Gamma(X) * I(X)$.*
3. *Let $pq < 0$ and $pq \neq -3$. Then there exist independent random variables ξ_1, ξ_2 with values in X and distributions μ_1, μ_2 such that the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, the distributions $\mu_j \notin \Gamma(X) * I(X)$, and each of the supports $\sigma(\mu_j)$ is not contained in a coset of a proper closed subgroup of the group X .*

To prove Theorem 2 we need some lemmas. The lemmas 7 and 8 given below were proved in [6] in the case $a = 1$. In the case $a \neq 1$ proofs of lemmas 7 and 8 follow the schemes of proofs of the corresponding lemmas in [6].

Lemma 7. *Let Y be an arbitrary Abelian group, let $a, b \in \mathbb{Z}$, $ab \neq 0$, and let $g_1(y)$ and $g_2(y)$ be functions on Y satisfying the equation*

$$g_1(u + av)g_2(u + bv) = g_1(u)g_1(av)g_2(u)g_2(bv), \quad u, v \in Y, \quad (21)$$

and the conditions

$$g_1(-y) = \overline{g_1(y)}, \quad g_2(-y) = \overline{g_2(y)}, \quad y \in Y, \quad g_1(0) = g_2(0) = 1. \quad (22)$$

Set $c = a - b$. If for certain $y_0 \in Y^{(c)}$ the inequality $g_1(y_0)g_2(y_0) \neq 0$ holds then there exists a subgroup $M = \{kabz_0\}_{k \in \mathbb{Z}}$ ($y_0 = cz_0$, $z_0 \in Y$) such that $g_1(y)g_2(y) \neq 0$ for $y \in M$.

Proof. Putting $u = -by$, $v = y$ and then $u = ay$, $v = -y$ in (21) we get

$$g_1(cy) = g_1(-by)g_1(ay)g_2(-by)g_2(by), \quad y \in Y, \quad (23)$$

$$g_2(cy) = g_1(ay)g_1(-ay)g_2(ay)g_2(-by), \quad y \in Y. \quad (24)$$

By the condition of the lemma $y_0 = cz_0$, $z_0 \in Y$. Substituting $y = z_0$ into (23) and (24) we conclude that

$$g_1(az_0) \neq 0, g_1(bz_0) \neq 0, \quad g_2(az_0) \neq 0, g_2(bz_0) \neq 0. \quad (25)$$

Putting $u = az_0$, $v = kz_0$ and then $u = bz_0$, $v = kz_0$, $k \in \mathbb{Z}$, in equation (21) we obtain

$$g_1((k+1)az_0)g_2((bk+a)z_0) = g_1(az_0)g_1(akz_0)g_2(az_0)g_2(bkz_0), \quad (26)$$

$$g_1((ak+b)z_0)g_2((k+1)bz_0) = g_1(bz_0)g_1(akz_0)g_2(bz_0)g_2(bkz_0). \quad (27)$$

Taking into account (25), it follows by induction from (26) and (27) that $g_1(kaz_0) \neq 0$, $g_2(kbz_0) \neq 0$, $k \in \mathbb{Z}$. The subgroup $M = \{kabz_0\}_{k \in \mathbb{Z}}$ is the required one.

Lemma 8. *Let M be an arbitrary subgroup in \mathbb{Q} , $g_1(y)$ and $g_2(y)$ be functions on M satisfying equation (21), conditions (22), and the conditions*

$$0 < g_1(y) \leq 1, \quad 0 < g_2(y) \leq 1. \quad (28)$$

Put $c = b - a$. Then on the subgroup $M^{(cab)}$ the following representation holds:

$$g_2(y) = \exp\{-\lambda_1 y^2\}, \quad g_2(y) = \exp\{-\lambda_2 y^2\}, \quad (29)$$

where $\lambda_j \geq 0$.

Proof. Set $\varphi_1(y) = -\ln g_1(y)$, $\varphi_2(y) = -\ln g_2(y)$. It follows from (21) that

$$\varphi_1(u+av) + \varphi_2(u+bv) = A(u) + B(v), \quad u, v \in M, \quad (30)$$

where $A(u) = \varphi_1(u) + \varphi_2(u)$, $B(v) = \varphi_1(av) + \varphi_2(bv)$.

We use the finite difference method to solve equation (30).

Let h_1 be an arbitrary element of M . Substitute $u + bh_1$ for u and $v - h_1$ for v in equation (30) and subtract equation (30) from the resulting equation. We get

$$\Delta_{ch_1}\varphi_1(u+av) = \Delta_{bh_1}A(u) + \Delta_{-h_1}B(v). \quad (31)$$

Putting $v = 0$ in (31) and subtracting the resulting equation from (31) we obtain

$$\Delta_{av}\Delta_{ch_1}\varphi_1(u) = \Delta_{-h_1}B(v) - \Delta_{-h_1}B(0). \quad (32)$$

Substitute $u + h_1$ for u in equation (32) and subtract equation (32) from the resulting equation. We get

$$\Delta_{h_1}\Delta_{av}\Delta_{ch_1}\varphi_1(u) = 0. \quad (33)$$

We conclude from (33) that the function $\varphi_1(y)$ satisfies the equation

$$\Delta_h^3 \varphi_1(y) = 0, \quad y \in M, h \in M^{(ca)}. \quad (34)$$

Reasoning similarly we get

$$\Delta_h^3 \varphi_2(y) = 0, \quad y \in M, h \in M^{(cb)}. \quad (35)$$

It follows from (34) and (35) that the functions $\varphi_j(y)$ are polynomials of the degree 2 on the subgroup $M^{(cab)}$. Taking into account (22) and (28) on the subgroup $M^{(cab)}$ we get $\varphi_j(y) = \lambda_j y^2$ where $\lambda_j \geq 0$.

Proof of Theorem 2. Let $pq > 0$. Lemma 6 implies that the linear forms $L'_1 = (p+q)\xi_1 + 2q\xi_2$ and $L'_2 = 2p\xi_1 + (p+q)\xi_2$ are independent. Making the substitution $\xi'_1 = (p+q)\xi_1$ and $\xi'_2 = 2q\xi_2$, we obtain that the linear forms $L'_1 = \xi'_1 + \xi'_2$ and $L'_2 = \frac{2p}{p+q}\xi'_1 + \frac{p+q}{2q}\xi'_2$ are also independent. We also note that if $\delta \in \text{Aut}(X)$ then the linear forms L_1 and L_2 are independent if and only if the linear forms L_1 and δL_2 are independent. Thus we may assume without loss of generality that $L'_1 = \xi'_1 + \xi'_2$ and $L'_2 = 4pq\xi'_1 + (p+q)^2\xi'_2$. Denote by μ'_j the distributions of random variables ξ'_j . Since $f_2, f_p, f_q, f_{p+q} \in \text{Aut}(X)$, if we prove that $\mu'_j = m_X$ then Theorem 2 in case 1 will be proved.

By Lemma 5 the independence of L'_1 and L'_2 implies that the characteristic functions of distributions μ'_j satisfy equation (6) which takes the form

$$\hat{\mu}'_1(u + 4pqv)\hat{\mu}'_2(u + (p+q)^2v) = \hat{\mu}'_1(u)\hat{\mu}'_1(4pqv)\hat{\mu}'_2(u)\hat{\mu}'_2((p+q)^2v), \quad u, v \in Y. \quad (36)$$

It is clear that the characteristic functions of distributions $\bar{\mu}'_j$ also satisfy equation (36). Therefore the characteristic functions of distributions $\nu_j = \mu'_j * \bar{\mu}'_j$ satisfy equation (36). Note that $\hat{\nu}_j(y) = |\hat{\mu}'_j(y)|^2 \geq 0$, $j = 1, 2$. We also note that since at least one support $\sigma(\mu_j)$ is not contained in a coset of a proper closed subgroup of the group X , we have that at least one support $\sigma(\nu_j)$ is not contained in a coset of a proper closed subgroup of the group X . It follows from (20) that at least for one j the equality

$$\{y \in Y : \hat{\nu}_j(y) = 1\} = \{0\} \quad (37)$$

holds.

Putting $u = -(p+q)^2y, v = y$ and then $u = -4pqy, v = y$ into equation (36) we obtain

$$\hat{\nu}_1((p-q)^2y) = \hat{\nu}_1((p+q)^2y)\hat{\nu}_1(4pqy)\hat{\nu}_2^2((p+q)^2y), \quad y \in Y. \quad (38)$$

$$\hat{\nu}_2((p-q)^2y) = \hat{\nu}_1(4pqy)\hat{\nu}_2(4pqy)\hat{\nu}_2((p+q)^2y), \quad y \in Y. \quad (39)$$

Assume first that $\hat{\nu}_1(y)\hat{\nu}_2(y) = 0$, $y \in Y, y \neq 0$. It follows from (38) that $\hat{\nu}_1((p-q)^2y) = 0$, $y \in Y, y \neq 0$. Since $f_{p-q} \in \text{Aut}(X)$, we conclude that $\hat{\nu}_1(y) = 0$, $y \in Y, y \neq 0$. Hence $\nu_1 = m_X$, so that $\mu'_1 = m_X$. Similarly, (39) implies that $\mu'_2 = m_X$.

Assume now that $\hat{\nu}_1(y_0)\hat{\nu}_2(y_0) \neq 0$ for some $y_0 \in Y, y_0 \neq 0$. Since $f_{p-q} \in \text{Aut}(X)$, we have that $Y^{((p-q)^2)} = Y$. We can apply Lemma 7 and obtain a subgroup $M \subset Y$ such that $\hat{\nu}_1(y)\hat{\nu}_2(y) \neq 0$. By Lemma 8 the restrictions of the characteristic functions $\hat{\nu}_1(y)$ and $\hat{\nu}_2(y)$ to $M^{(4pq(p+q)^2)(p-q)^2)}$ have form (29). Substituting these representations into (36) we get

$$4pq\lambda_1 + (p+q)^2\lambda_2 = 0.$$

Since $pq > 0$, this equality implies that $\lambda_1 = \lambda_2 = 0$. Hence $\hat{\nu}_1(y) = \hat{\nu}_2(y) = 1$ for $y \in M^{(4pq(p+q)^2)(p-q)^2}$, that contradicts (37).

Let $pq = -3$. The desired statement follows from statement 1 of Theorem 1.

Let $pq < 0$, $pq \neq -3$. Denote by ω_j the distributions constructed in the proof of Theorem 1 in corresponded cases. We note that the characteristic functions $\hat{\omega}_1(y)$ and $\hat{\omega}_2(y)$ satisfy equation (12). Denote by γ_j Gaussian distributions on X with the characteristic functions $\hat{\gamma}_1(y) = e^{-\lambda y^2}$, $\hat{\gamma}_2(y) = e^{\frac{p}{q}\lambda y^2}$, where $\lambda > 0$. It is easy to verify that the functions $\hat{\gamma}_1(y)$ and $\hat{\gamma}_2(y)$ satisfy equation (12). Put $\mu_j = \omega_j * \gamma_j$. It is obvious that the functions $\hat{\mu}_1(y)$ and $\hat{\mu}_2(y)$ satisfy equation (12). Since a support of a symmetric non degenerate Gausssian distribution is a connected subgroup, we have that $\sigma(\gamma_j) = X$. Hence $\sigma(\mu_j) = X$. By the construction $\mu_j \notin \Gamma(X) * I(X)$. Thus μ_j are the desired distributions.

Remark 5. The example of distributions constructed in Remark 2 shows that statement 2 of Theorem 2 may not be strengthened to the statement that both distributions $\mu_j \in \Gamma(X) * I(X)$.

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